Set Theory

(see http://en.wikipedia.org/wiki/Naive_set_theory)

Sets

We use the term **Set** in mathematics to refers to a group of objects. We can show a set by listing all or part of its members, for example

A = { 1, 3, 5, 16 }
B = { 0, 5, 10, 15, 20, 25, . . . }
ℕ = { 0, 1, 2, 3, 4, 5, 6, . . . } (Natural numbers)

We can also describe a set according to the properties of its elements, for example

A = { x : x ∈ ℕ and x is even }

This should be read as, "A is the set of all elements **x** such that **x** is an element of the natural numbers and **x** is even". This is the set of all even integers greater than or equal to zero.

Here are symbols for other commonly used sets in mathematics.

∅ = { } (The empty set or the set with no elements)
ℕ = { 0, 1, 2, 3, 4, 5, 6, . . . } (Natural numbers)
Z = { . . ., -3, -2, -1, 0, 1, 2, 3, . . . } (Integers)
Q = the set of all rational numbers (Q stands for quotient)
ℝ = the set of all real numbers
= the set of all imaginary numbers
ℂ = the set of all complex numbers

It is sometimes convenient to show sets and their subsets as Venn diagrams, which is a rectangle representing the universe, with sets inside the universe represented as circles.

Subsets

In the real world, we often want to refer to a group of elements that are contained within or part of an already existing set. This group of elements is actually a set made up of elements from the original set, and is called a **subset**. Consider the two sets below. All of the items in set T are also contained in set S; therefore, set T is a subset of set S.

S = {1, 2, 3, 4, 5, 6, 7, 8)
T = {2, 4, 6, 8}

The symbol used to denote a subset is **⊆**, and the subset in the example above would be denoted as $T \subseteq S$.

The formal definition of a subset is defined below and can be read as, "T is a subset of all elements **x** such that every **x** is an element of set S.

$$T = \{ x : x \in S \}$$

Set Equality

Two sets are equal (T = S), if and only if every element in T is in S and every element in S is in T. It is denoted as:

$$T \subseteq S \text{ and } S \subseteq T$$

Proper Subsets

T is a proper subset of S if every element of T belongs to S, and T is not equal to S, or in other words $T \subseteq S$ and $T \neq S$. A proper subset is denoted as:

$$T \subset S$$

Superset

If T is a subset of S ($T \subseteq S$) then we can refer to set S as the **superset** of T. A superset is denoted as:

$$S \supseteq T$$

Set relationships

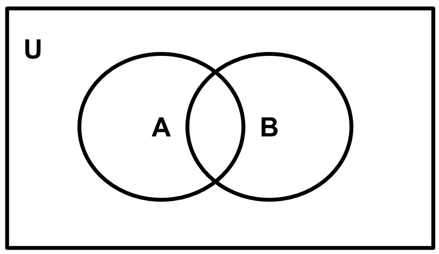
Two sets often have unique relationships that are useful to in every day life. For example, "I want to find all of the students that are in both class A and B", or "I want to find every body that is in class A that is not also in class B. The following terms refer to commonly used set relationships: union, intersection, relative complement, symmetric difference, universal complement, and disjoint.

Union

The union of two sets is useful for combining two sets of elements together into one set. The union gives all of the unique elements found in either set A or set B. The union of two sets is defined as:

$$A \cup B = \{ x : x \in A \text{ or } x \in B \text{ or both} \}$$

Draw the Venn diagram of the union of two sets below:

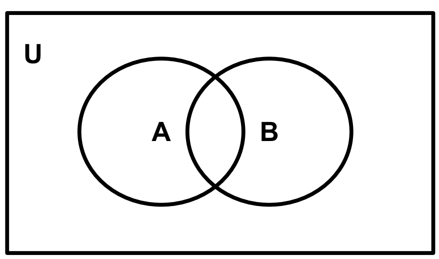


Intersection

The **intersection** of two sets is useful for describing the group of elements that are common to two sets. The intersection gives all of the unique elements found in either set A and in set B. The intersection of two sets is defined as:

$$A \cap B = \{ x : x \in A \text{ and } x \in B \}$$

Draw the Venn diagram for the intersection of two sets below:

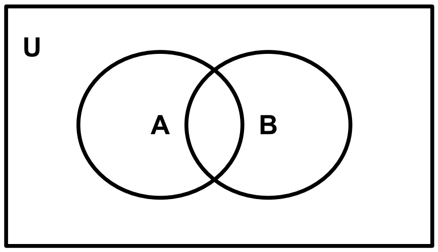


Relative complement

The **relative complement** of two sets is useful for finding the group of elements that are not in the other set. The relative complement describes all of the unique elements found in set A but not in set B. The relative complement of two sets is defined as:

$$A \setminus B = \{ x : x \in A \text{ and } x \notin B \}$$

Draw the Venn diagram for the relative complement of two sets below:



Symmetric difference

The symmetric difference of two sets gives all the elements that are not common between two groups or sets. The symmetric difference describes all of the unique elements found in set A and in set B but not in both. The symmetric difference of two sets is defined as:

$$A \oplus B = \{ x : x \in A \text{ or } x \in B \text{ but not both } \}$$

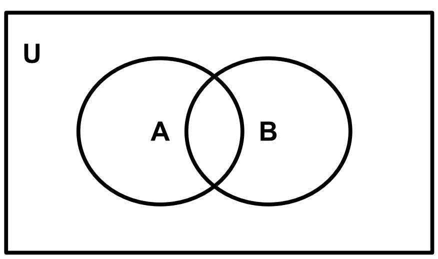
or

$$A \oplus B = (A \cup B) \setminus (A \cap B)$$

or

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Draw the Venn diagram for the symmetric difference of two sets below:



Universal Complement

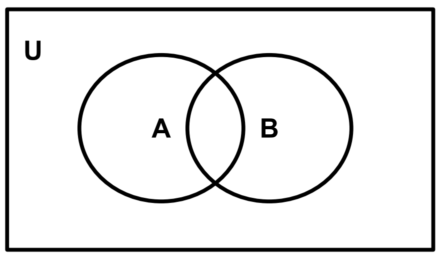
The **universal complement** of a set is used to refer to everything that is not in the set.

To understand the universal complement we need to define the term **universe** as it is used in set theory. The term universe refers to all of the possible elements. This includes all of the elements in set A and B and all other possible elements.   For example, the the entire student body can be thought of the universe of all students. A some of students in the universe (the entire student body) are in class A and some of the students are in class B.

The universal complement describes all of the elements in the universe that are not in set A or set B. The universal complement of two sets is defined as:

$A^c = U \setminus A = \{ x : x \notin A \}$

Draw the Venn diagram for the universal complement of two sets below:

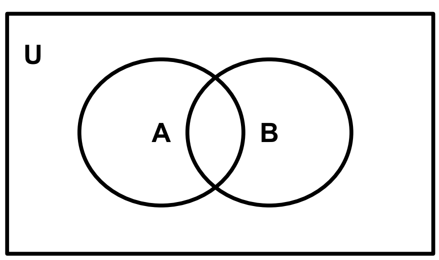


Disjoint

The term **disjoint** means that of two sets have nothing in common. The universal complement of two sets is defined as:

$A \cap B = \varnothing$ (empty set)

Draw the Venn diagram for disjoint sets below:



Common Set Operations

(see http://en.wikipedia.org/wiki/Algebra_of_sets)

1a.	$A \cup B = B \cup A$	commutative laws
b.	$A \cap B = B \cap A$	
2a.	$(A \cup B) \cup C = A \cup (B \cup C)$	associative laws
b.	$(A \cap B) \cap C = A \cap (B \cap C)$	
3a.	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	distributive laws
b.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
4a.	$A \cup A = A$	idempotent laws
b.	$A \cap A = A$	
5a.	$A \cup \varnothing = A$	identity laws
b.	$A \cup U = U$	
c.	$A \cap \varnothing = \varnothing$	
d.	$A \cap U = A$	
6a.	$A \cup A^c = U$	complement laws
b.	$A \cap A^c = \varnothing$	
7.	$(A^c)^c = A$	double complement
8a.	$U^c = \varnothing$	
b.	$\varnothing^c = U$	
9a.	$(A \cup B)^c = A^c \cap B^c$	DeMorgan's laws
b.	$(A \cap B)^c = A^c \cup B^c$	

Examples of Set Operations

Let the universe U = { q, r, s, t, u, v, w, x, y, z }, A={q,r,s,t}, and B={s,t,u,v}

$A \cup B = \{ q, r, s, t, u, v \}$
$A \cap B = \{ s, t \}$
$A \setminus B = \{ q, r \}$
$B \setminus A = \{ u, v \}$
$A \oplus B = \{ q, r, u, v \}$
$B \oplus A = \{ q, r, u, v \}$

$A^c = U \setminus A = \{ u, v, w, x, y, z \}$
$B^c = U \setminus B = \{ q, r, w, x, y, z \}$
$(A \cup B)^c = U \setminus (A \cup B) = \{ w, x, y, z \}$
$(A \cap B)^c = U \setminus (A \cap B) = \{ q, r, u, v, w, x, y, z \}$

$A^c \cup B = \{ s, t, u, v, w, x, y, z \}$
$A^c \cap B = \{ u, v \}$

Set Theory Worksheet

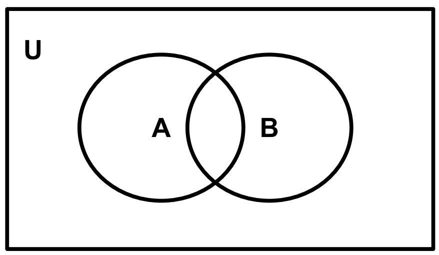
Give informal and formal descriptions, and draw the Venn diagram for each of the following set relationships.

Union

Informal description:

Formal description:

Venn diagram:

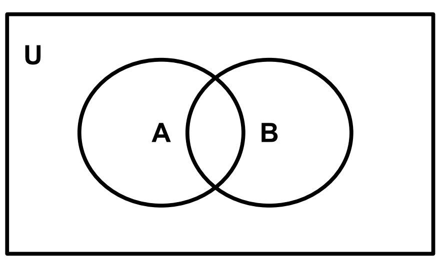


Intersection

Informal description:

Formal description:

Venn diagram:

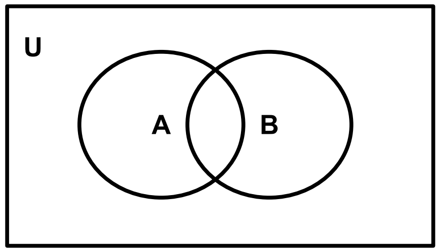


Relative Complement

Informal description:

Formal description:

Venn diagram:

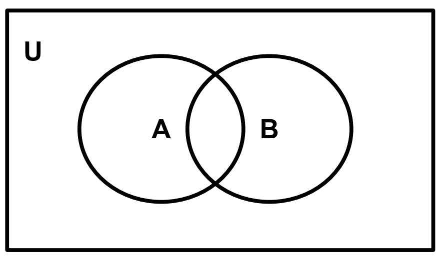


Symmetric Difference

Informal description:

Formal description:

Venn diagram:

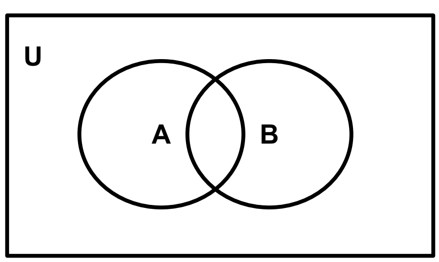


Universal Complement

Informal description:

Formal description:

Venn diagram:

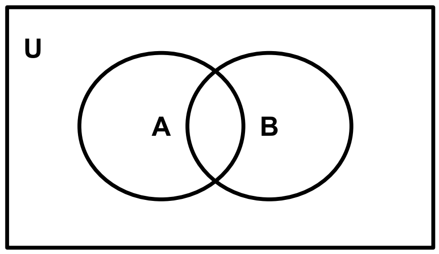


Disjoint

Informal description:

Formal description:

Venn diagram:



Solve the following set operations:

Let the universe U = { a, b, c, d, 1, 2, 3, 4 }, A = { b, c, d, 1 }, and B={,2,3 }

$A \cup B =$
$A \cap B =$
$A \setminus B =$
$B \setminus A =$
$A \oplus B =$
$B \oplus A =$
$A^c =$
$B^c =$
$(A \cup B)^c =$
$(A \cap B)^c =$
$A^c \cup B =$
$A^c \cap B =$

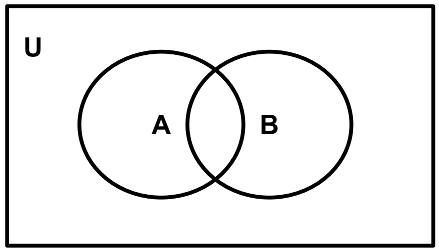
Let the universe U = { α, ρ, σ, π, λ, μ }, A = { α, ρ, λ }, and B={λ,μ}

$A \cup B =$
$A \cap B =$
$A \setminus B =$
$B \setminus A =$
$A \oplus B =$
$A^c =$
$B^c =$
$A^c \cup B^c =$
$A^c \cap B^c =$
$(A \setminus B)^c =$
$(A \oplus B)^c =$

Let the universe U={0,1,2,3,4,5,6,7,8,9}, A={1, 5, 7, 8}, and B={2,4,5,6}.

$A^c \cap B =$
For any set A, what is:
$A \cap A^c =$
$A \setminus A^{c=} =$
For any set Y, what is:
$Y \oplus Y =$
$Y \oplus \varnothing =$

On the Venn diagram below, show $A \cap B^c$



Use the Venn diagrams below to prove

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$